

Developing 6DT and directional time theory:

A framework and notes

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Abstract

This document collects concrete proposals, equations, and working definitions that address five essential tasks for developing the 6DT model ("vector time" in a six-dimensional manifold). The aim is practical: (i) propose an explicit set of constraints / gauge principles that remove negative-norm states while preserving the model's phenomenology; (ii) write the complete (reduced) 6D dynamical field equations and show how to set up symmetric solvers; (iii) outline quantization paths that maintain unitarity (or replace it with a controlled BRST/pseudo-unitary construction); (iv) translate the model into experimental parameter-space constraints and give recipes for data-driven bounds; (v) give a roadmap for classifying geometric compactifications and signature-preserving reductions compatible with physics. It then explores deeper mathematics in 4 different ways that directly relate the model to foundational physics.

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1 Overview and notation

We follow the notation of the parent manuscript: the 6D coordinates are $X^A = \{t^i, x^j\}$ with $i, j = 1, 2, 3$. The metric ansatz used in the body of the paper is

$$G_{AB}(X) = \begin{pmatrix} -c^2 \delta_{ij} & \epsilon K_{ik}(\mathbf{x}) \\ \epsilon K_{kj}(\mathbf{x}) & \delta_{kj} \end{pmatrix}, \quad (1)$$

where $K_{ij}(\mathbf{x})$ is a symmetric spatial tensor sourced by the Hessian of the Newtonian potential and $\epsilon \ll 1$ is a small dimensionless coupling. The sign convention is mostly plus for the purely spatial x^j block and mostly minus for the t^i block; the signature is $(-, -, -, +, +, +)$ in this basis.

We will use canonical (Hamiltonian) language where convenient. The worldline Lagrangian for

a massive test particle of mass m is

$$L = \frac{1}{2}m G_{AB}(X)\dot{X}^A\dot{X}^B, \quad \dot{X}^A = \frac{dX^A}{d\tau}. \quad (2)$$

Canonical momenta are $P_A = \partial L / \partial \dot{X}^A = mG_{AB}\dot{X}^B$.

This supplement assumes the reader is familiar with the derivations in the uploaded manuscript and aims to provide concrete technical material useful for follow-up calculations and implementation.

2 Task 1: Constraint / gauge structure

2.1 Goals and strategy

Extra time components induce negative-norm directions in the kinetic form. To remove unphysical (ghost) states while keeping a single observable effective time, one needs a set of first-class constraints that reduce the physical phase space. Two broad strategies are natural:

1. Adopt a *local gauge symmetry* acting on the internal three-dimensional time index i so that different t^i are gauge-equivalent; gauge-fixing reduces to one physical time direction.
2. Impose *first-class constraint functions* (à la Dirac) that eliminate the extra canonical degrees of freedom directly; quantization then imposes the constraints on the state space.

Both approaches are compatible and can be combined: introduce a gauge symmetry whose Noether generators coincide with the first-class constraints.

2.2 A concrete proposal: local $SO(3)_t$ gauge and normalization

A minimal and physically intuitive gauge is local rotations among the three time components, $SO(3)_t$. The idea: treat the vector-of-times $t^i(\tau)$ as a triplet on which a local $SO(3)$ acts; gauge-equivalence identifies all directions in the internal time space, leaving a single gauge-invariant “length” of the time flow.

Impose the *normalization constraint*

$$\Phi_0(X, P) \equiv \frac{1}{c^2} G_{ij}^{(t)} P_{ti} P_{tj} + m^2 c^2 \approx 0, \quad (3)$$

where $G_{ij}^{(t)} = -c^2 \delta_{ij}$ is the time-block of G_{AB} in the uncoupled limit (and P_{ti} are the canonical momenta conjugate to the t^i). Physically (3) enforces that the time-sector momenta sit on a single

mass-shell; classically it fixes the overall normalization of $\dot{t}^i$. Eq. (3) is analogous to the mass-shell constraint in ordinary relativistic particle mechanics; it is a first-class constraint generator of worldline reparameterizations.

Next introduce the three generators of internal rotations (the $SO(3)_t$ algebra):

$$J_{ij} \equiv t_i P_{tj} - t_j P_{ti}, \quad i, j \in \{1, 2, 3\}. \quad (4)$$

They satisfy $\{J_{ij}, J_{kl}\} = \delta_{jk} J_{il} - \delta_{ik} J_{jl} - \delta_{jl} J_{ik} + \delta_{il} J_{jk}$ and are *first-class* provided they have weakly vanishing Poisson brackets with the total Hamiltonian (after adding suitable constraints). Implementing $J_{ij} \approx 0$ as first-class constraints identifies all internal orientations of the time vector: physically one picks a gauge where the time vector points along a fixed axis in the internal space.

2.3 Counting and reduction

Phase-space counting: naive degrees of freedom for a worldline in 6D are $2 \times 6 = 12$. The constraints we impose are:

- the reparameterization (mass-shell) constraint Φ_0 (one first-class constraint);
- the three $SO(3)_t$ generators J_{ij} (three first-class constraints);
- optionally a set of three gauge-fixing conditions (one per generator) to fully fix the gauge.

Each first-class constraint removes two phase-space dimensions (one configuration + one momentum) after gauge fixing. Thus $(4 \text{ first-class}) \times 2 = 8$ reductions leave $12 - 8 = 4$ physical phase-space dimensions, corresponding to a single time and the usual three spatial momenta (or equivalently one time DOF + 3 spatial DOFs), matching physical expectations for a point particle.

2.4 Remarks on alternative algebras

The $SO(3)$ choice is not unique. Inspired by 2T physics (which uses $Sp(2, \mathbb{R})$) one could instead impose an algebra generated by quadratic constraints built from t^i and P_{ti} (e.g. $t^i t^i$, $t^i P_{ti}$, $P_{ti} P_{ti}$) that realize an $\mathfrak{sl}(2)$ -like algebra. For three times a natural larger algebra is $\mathfrak{sl}(3, \mathbb{R})$ or its compact real form; however larger noncompact algebras introduce technical complications in ensuring unitarity. Empirically, a compact internal gauge like $SO(3)$ is simplest for removing orientation degrees of freedom while maintaining a positive-definite elimination of ghosts.

3 Task 2: Complete 6D dynamics

3.1 Field equations: Einstein-type dynamics

To promote the metric ansatz (1) to dynamical fields we write an Einstein–Hilbert action in six dimensions:

$$S_{\text{grav}} = \frac{1}{2\kappa_6^2} \int d^6 X \sqrt{|G|} R^{(6)}[G], \quad (5)$$

with $\kappa_6^2 = 8\pi G_6$. Variation yields

$$R_{AB}^{(6)} - \frac{1}{2} G_{AB} R^{(6)} = \kappa_6^2 T_{AB}^{(6)}. \quad (6)$$

We take a conservative approach: do not allow independent sources in the t^i directions (both to remain phenomenologically safe and to keep minimal coupling). For the application considered in the parent manuscript, it is consistent to take $T_{AB}^{(6)}$ to have support only on the 3D spatial slices and to vanish along pure time–time components, so that the time- sector dynamics are driven by geometry and by the spatial matter distribution (i.e. K_{ij} remains sourced by spatial mass moments).

3.2 Reduced equations for the ansatz

Plugging the block ansatz (1) into the Einstein tensor and expanding in powers of ϵ yields a useful perturbative system. To linear order in ϵ one finds schematically (index placement suppressed):

$$G_{tt}^{(6)} = G_{ij}^{(t)} + \epsilon \mathcal{D}_{ij}[K] + \mathcal{O}(\epsilon^2), \quad (7)$$

$$G_{tx}^{(6)} = \epsilon \mathcal{L}_{ij}[\partial K] + \mathcal{O}(\epsilon^2), \quad (8)$$

$$G_{xx}^{(6)} = G_{ij}^{(x)} + \epsilon \mathcal{M}_{ij}[K] + \mathcal{O}(\epsilon^2), \quad (9)$$

where $\mathcal{D}, \mathcal{L}, \mathcal{M}$ are differential operators acting on $K_{ij}(\mathbf{x})$ built from spatial derivatives and the flat-space connection. The explicit components are straightforward but lengthy; for practical numerical work it is best to derive them using a computer algebra system (e.g. `xAct` in Mathematica or similar). The important point is that to leading order the Einstein equations become a linear (or weakly nonlinear) elliptic/Poisson- like system for K_{ij} sourced by spatial matter moments.

3.3 Spherically symmetric mass: a worked reduction

For a point mass at the origin the Newtonian potential has the form $\Phi(r) = -GM/r$. Using the ansatz $K_{ij} = c^{-2}\partial_i\partial_j\Phi$, inserted into Eq. (6) one can check consistency at $\mathcal{O}(\epsilon)$ by verifying the tx and xx components are satisfied by the background plus the sourced K_{ij} . In practice the reduced system for $K(r)$ in spherical symmetry collapses to ordinary radial ODEs. This provides a practical consistency check: solve the linearized equations for $K_{ij}(r)$ and confirm $\epsilon|K| < c^2$ everywhere (signature-preserving condition).

3.4 Numerical strategy for more general symmetric cases

For axisymmetric or stationary systems (rotating oblate planets), reduce to 2D elliptic PDEs in (r, θ) for the nonzero components of K_{ij} . A stable numerical pipeline is:

1. Expand $K_{ij}(r, \theta) = \sum_{\ell} K_{ij}^{(\ell)}(r) P_{\ell}(\cos \theta)$ and truncate at finite $\ell_{\max}$ (spherical harmonic decomposition).
2. For each mode solve the radial ODE / boundary-value problem using spectral shooting or finite-difference relaxation; impose regularity at the origin and decay at infinity.
3. Reconstruct $K_{ij}(r, \theta)$ and compute check-quantities (e.g. eigenvalues of G_{AB} at each grid point) to ensure no signature flips.

This strategy is directly implementable in standard PDE libraries (SUNDIALS, PETSc) or spectral packages.

4 Task 3: Quantization

4.1 Overview of obstacles

Quantization must address two intertwined issues: (i) extra timelike directions typically produce negative-norm (ghost) excitations and (ii) the lack of compactness in time directions complicates Kaluza–Klein intuition. The constraint/gauge structure of Section 2 aims to remove unphysical degrees classically; quantization must implement these constraints at the operator or path-integral level.

4.2 Canonical quantization with first-class constraints

Adopt Dirac quantization for constrained systems.

- Promote canonical variables to operators with $[X^A, P_B] = i\hbar \delta_B^A$.
- Impose first-class constraints $\hat{\Phi}_\alpha |\psi\rangle = 0$ for the set $\{\text{mass-shell}, J_{ij}, \dots\}$ chosen in Task 1. Physical Hilbert space is the kernel of all constraint operators modulo null states.

If the constraints close with operator-ordering anomalies, one adds suitable quantum corrections or modifies the operator ordering until anomaly-free. This is the standard strategy in e.g. 2T physics.

4.3 BRST quantization (preferred practical method)

BRST provides an elegant and practical route. For a set of first-class constraints Φ_α with algebra $\{\Phi_\alpha, \Phi_\beta\} = f_{\alpha\beta}^\gamma \Phi_\gamma$ define ghosts c^α and conjugate antighosts b_α and the nilpotent BRST charge

$$Q_{\text{BRST}} = c^\alpha \Phi_\alpha - \frac{1}{2} f_{\alpha\beta}^\gamma c^\alpha c^\beta b_\gamma + \dots \quad (10)$$

Physical states satisfy

$$Q_{\text{BRST}} |\Psi\rangle = 0, \quad |\Psi\rangle \sim |\Psi\rangle + Q_{\text{BRST}} |\Lambda\rangle. \quad (11)$$

The BRST cohomology automatically quotients out gauge-equivalent and negative-norm states coming from pure-gauge excitations, leading (when done correctly) to a positive-definite physical inner product. Implementing this for the $SO(3)_t$ plus reparameterization algebra is standard: the ghosts are three $SO(3)$ ghosts plus the reparameterization ghost.

4.4 Path-integral representation with gauge fixing

A practical path-integral useful for perturbation theory about a classical background is

$$Z = \int [dG][d\Phi d\bar{c} dc] \delta(\chi) \Delta_{\text{FP}}[G] e^{\frac{i}{\hbar} S_{\text{tot}}[G, \Phi, c, \bar{c}]}, \quad (12)$$

where χ are gauge-fixing conditions (e.g. fix orientation of t^i and proper-time gauge), Δ_{FP} the Faddeev–Popov determinant implemented by ghosts, and S_{tot} includes the gravitational action, matter action and the ghost action. Correlation functions of BRST-invariant operators computed from (12) lie in the physical cohomology.

4.5 Comments on pseudo-unitary approaches

If one wants to explore alternative quantization that does not entirely eliminate extra time excitations, pseudo-unitary (indefinite inner product) frameworks are available. They can be made consistent as effective theories for restricted energy ranges, but are uglier to interpret. We therefore recommend the BRST-first-class-constraint path as the principled route.

5 Task 4: Phenomenological bounds and data pipelines

5.1 Mapping to SME-style effective coefficients

The paper already shows a linear identification

$$c_{ij}^{(\text{eff})}(\mathbf{x}) \simeq \epsilon K_{ij}(\mathbf{x}). \quad (13)$$

For a given spatial location (e.g. Earth surface, spacecraft trajectory) compute the tidal tensor $K_{ij}(\mathbf{x})$ from the known mass distribution and form the effective anisotropic coefficients entering dispersion relations and nonrelativistic Hamiltonians.

5.2 Order-of-magnitude sensitivity formulas

Below are simple back-of-envelope formulas that translate an experimental null result into a bound on ϵ (following the estimations in the parent manuscript). Consider three representative classes of experiments.

Astrophysical time-of-flight (GRB/FRB) A direction-dependent fractional speed shift of order $\sim \epsilon(T^T K n)$ accumulates across distance D , producing a differential time delay between two photon energies or arrival angles of

$$\Delta t_{\text{LIV}} \sim \frac{D}{c} \epsilon |T^T K n|. \quad (14)$$

A null measurement $|\Delta t_{\text{LIV}}| < \Delta t_{\text{obs}}$ implies

$$|\epsilon| \lesssim \frac{c \Delta t_{\text{obs}}}{D |K|_{\text{eff}}}. \quad (15)$$

Terrestrial clocks (Hughes–Drever type) Nonrelativistic energy shifts scale as $\delta E \sim \epsilon K_{ij} Q^{ij}$ where Q^{ij} is an experiment-dependent tensor (spin or quadrupole expectation). A sensitivity δE_{min}

yields

$$|\epsilon| \lesssim \frac{\delta E_{\min}}{|K||Q|}. \quad (16)$$

Shapiro / Solar System tests Directional corrections to light travel time near a massive body are

$$\Delta t_{\text{dir}} \sim \frac{\epsilon}{c} \int_{\text{path}} K_{ij} n^i n^j dl. \quad (17)$$

A null constraint $|\Delta t_{\text{dir}}| < \Delta t_{\text{exp}}$ yields a bound analogous to (14) but with D replaced by the effective near-limb path length and K the limb value.

5.3 Recommended data analysis pipeline

To obtain robust bounds one should perform a global fit (Bayesian or frequentist) of the effective coefficients built from $K_{ij}(\mathbf{x}; \theta)$ (where θ parameterizes e.g. ϵ , alignment angles, or extra model parameters) against datasets such as: GRB/FRB time-of-flight catalogs, pulsar timing arrays, atomic-clock sidereal-modulation data, spacecraft (Cassini-like) time-delay residuals. Steps:

1. Build a forward model that computes the predicted signal for given ϵ and known mass distributions (use precise Earth/solar multipoles).
2. For terrestrial tests, include laboratory orientation and time-tagging to model sidereal modulations.
3. Define a joint likelihood and run an MCMC or nested-sampling pipeline to map posterior constraints on ϵ and nuisance parameters.

This pipeline directly produces statistically rigorous bounds and handles correlations between instruments and datasets.

6 Task 5: Mathematical classification of geometries

6.1 Scope and desirable properties

We seek classifications of 6D manifolds $(\mathcal{M}^6, G)$ with signature $(3, 3)$ (or equivalently three time-like + three space-like directions) that admit metric ansatzes of the form used in the 6DT model and that

preserve physical acceptability (no signature flips, hyperbolicity of wave equations in the physical sector, etc.). Properties of interest include:

- existence of an integrable 3D time-foliation or an $SO(3)$ principal bundle structure over a 3D spatial base;
- holonomy restrictions that avoid exotic causal behaviour (e.g. require holonomy contained in $SO^+(3, 3)$ with additional reductions);
- compatibility with (para-)quaternionic or product-structure reductions that permit a controlled dimensional reduction to an effective 4D theory.

6.2 Concrete classification strategies

A few interlocking mathematical programs are recommended:

1. G-structure / holonomy analysis Classify manifolds admitting a reduction of the frame bundle to subgroups that tame the metric signature. For example, investigate reductions to $SO(3) \times SO(3)$ (product structure), $GL(3, \mathbb{R})$ (general linear reductions), or para-quaternionic groups (which are natural in neutral-signature contexts). Compute intrinsic torsion classes to determine possible parallel spinors and preserved supersymmetry-like structures (if a supersymmetric extension is desired).

2. Fibration and bundle reductions View the 6D manifold as a (possibly twisted) fibration with 3D fibre F_t^3 carrying the internal time structure and 3D base B_x^3 the spatial manifold. Classification reduces to classifying principal bundles $F_t^3 \rightarrow \mathcal{M}^6 \rightarrow B_x^3$ with structure group $\mathcal{G}$ (e.g. $SO(3)_t$). Obstructions to trivialization are topological invariants (e.g. Stiefel–Whitney classes) which can be enumerated.

3. Signature-preserving compactifications If one considers compactifying or quotienting the internal time manifold (dangerous but mathematically interesting), restrict to quotients that preserve local hyperbolicity for the physical modes; natural candidates include compact nilmanifolds or tori with appropriately chosen indefinite metrics. Para-complex and para-quaternionic compactifications are also worth investigating because they generalize Kähler/quaternionic geometries to neutral signatures.

6.3 Examples and minimal models

Provide a short list of tractable model spaces for exploratory calculations:

- *Product model*: $\mathcal{M}^6 = \mathbb{R}_t^3 \times \mathbb{R}_x^3$ with metric of the ansatz type and K_{ij} determined by external matter — simplest toy.
- *Twisted $SO(3)$ -bundle*: nontrivial principal $SO(3)$ bundle over S^3 or a compact 3-manifold; study small torsion classes and their effect on wave propagation.
- *Para-quaternionic model*: local charts with para-quaternionic structure that realize a neutral signature and allow controlled reductions.

Each of these models can be used to track mode spectra, compute holonomy, and test stability.

7 Worked appendices

7.1 Appendix A: Canonical expressions for the particle momenta

From (2) the canonical momenta for the worldline are explicitly

$$P_{t^i} = m \left(-c^2 \dot{t}^i + \epsilon K_{ij}(\mathbf{x}) \dot{x}^j \right), \quad (18)$$

$$P_{x^j} = m \left(\dot{x}^j + \epsilon K_{ij}(\mathbf{x}) \dot{t}^i \right). \quad (19)$$

These relations can be inverted perturbatively in ϵ to express velocities in terms of momenta, which is useful for deriving a Hamiltonian and identifying primary constraints.

7.2 Appendix B: Sample BRST transformations

For the gauge algebra generated by $\{\Phi_0, J_{ij}\}$ introduce ghosts c_0 and $c^{ij} = -c^{ji}$ with antighosts b^0, b_{ij} . The nilpotent BRST transformations (on phase space variables) take the schematic form

$$\delta_{\text{BRST}} X^A = \{X^A, \Phi_\alpha\} c^\alpha, \quad (20)$$

$$\delta_{\text{BRST}} P_A = \{P_A, \Phi_\alpha\} c^\alpha, \quad (21)$$

$$\delta_{\text{BRST}} c^\alpha = -\frac{1}{2} f_{\beta\gamma}^\alpha c^\beta c^\gamma, \quad (22)$$

$$\delta_{\text{BRST}} b_\alpha = B_\alpha, \quad (23)$$

where B_α are Nakanishi–Lautrup auxiliary fields implementing gauge fixing.

8 Symbolic Implementation of the Constraint Algebra and BRST Formalism

This section details the procedure for implementing the proposed gauge structure in a computer algebra system (CAS) to verify its classical and quantum consistency. The primary goal is to construct the Becchi-Rouet-Stora-Tyutin (BRST) charge and rigorously test its nilpotency, which is the cornerstone of a consistent quantum gauge theory.[4]

8.1 Phase Space Representation and Poisson Brackets

The foundation of any canonical analysis is the Hamiltonian framework defined on the system's phase space. For the 6DT model describing a point particle, the phase space is a 12-dimensional manifold. The first step in a symbolic implementation is to represent this structure.

In a CAS such as Mathematica, SymPy, or Maple, one defines the 6D phase space coordinates $X^A = \{t^i, x^j\}$ and their conjugate momenta $P_A = \{P_{t^i}, P_{x^j}\}$, for $i, j \in \{1, 2, 3\}$, as symbolic variables.[6, 7] These are treated as independent functions of a worldline parameter τ . The core dynamical structure is encoded in the fundamental Poisson bracket, which is implemented as a symbolic differential operator. For any two functions $f(X, P)$ and $g(X, P)$ on the phase space, the Poisson bracket is defined as [8]:

$$\{f, g\} \equiv \sum_{A=0}^5 \left(\frac{\partial f}{\partial X^A} \frac{\partial g}{\partial P_A} - \frac{\partial f}{\partial P_A} \frac{\partial g}{\partial X^A} \right)$$

This operator must be programmed to correctly handle symbolic differentiation and simplification. This setup provides a complete symbolic representation of a constrained Hamiltonian system, a structure that is central to the formulation of gauge theories and general relativity.[9] All subsequent algebraic checks rely on the correct implementation of this fundamental bracket.

8.2 The Classical Constraint Algebra

The 6DT model introduces unphysical degrees of freedom associated with the extra time dimensions, which manifest as negative-norm states or "ghosts" in a quantum theory. To eliminate these, a set of first-class constraints is imposed. A constraint is first-class if its Poisson bracket with all

other constraints vanishes weakly (i.e., vanishes on the surface in phase space where the constraints themselves are satisfied).[9] The verification of this property is a critical consistency check of the classical theory.

The proposed constraints are defined symbolically within the CAS:

1. **Mass-Shell Constraint:** This constraint, Φ_0 , generates worldline reparameterizations and is analogous to the standard relativistic mass-shell condition. In the uncoupled limit, it is given by:

$$\Phi_0 \equiv \frac{1}{c^2} G_{ij}^{(t)} P_{ti} P_{tj} + m^2 c^2 = -P_{ti} P_{ti} + m^2 c^2 \approx 0$$

2. **Gauge Constraints:** These three constraints, J_{ij} , are the generators of the proposed local $SO(3)_t$ gauge symmetry, which acts as rotations on the internal time-vector indices. They are defined as:

$$J_{ij} \equiv t_i P_{tj} - t_j P_{ti} \approx 0 \quad (\text{for } i < j)$$

These are the standard generators of rotations in the $\{t^i, P_{ti}\}$ subspace of the full phase space.[8]

Using the symbolic Poisson bracket operator defined in the previous section, the full constraint algebra is computed. The verification process involves three distinct sets of calculations:

- **Closure of the Gauge Group:** The brackets between the J_{ij} generators must close to form the $\mathfrak{so}(3)$ Lie algebra. The symbolic computation must reproduce the well-known result:

$$\{J_{ij}, J_{kl}\} = \delta_{jk} J_{il} - \delta_{ik} J_{jl} - \delta_{jl} J_{ik} + \delta_{il} J_{jk}$$

A failure to reproduce this result would indicate an error in the implementation of the bracket or the definitions of the constraints.

- **Invariance of the Mass-Shell:** The mass-shell constraint Φ_0 must be invariant under the $SO(3)_t$ gauge transformations. This requires that its Poisson bracket with all gauge generators vanishes. The symbolic calculation must confirm:

$$\{\Phi_0, J_{ij}\} = 0$$

A non-zero result would signal a fundamental inconsistency, as the physical mass-shell condition would not be respected by the proposed gauge symmetry.

- **Self-Commutativity:** The mass-shell constraint must Poisson-commute with itself, which is trivially true: $\{\Phi_0, \Phi_0\} = 0$.

Successful symbolic verification of these relations confirms that the set of constraints $\Phi_\alpha = \{\Phi_0, J_{12}, J_{23}, J_{31}\}$ forms a closed, first-class algebra. The structure constants $f_{\alpha\beta}^\gamma$, defined by $\{\Phi_\alpha, \Phi_\beta\} = f_{\alpha\beta}^\gamma \Phi_\gamma$, are non-zero only for the $\mathfrak{so}(3)$ part of the algebra. This classical consistency is a prerequisite for constructing a well-defined BRST charge.

8.3 Construction and Verification of the BRST Charge

The BRST formalism provides a powerful cohomological method for quantizing gauge theories, ensuring that unphysical states are systematically removed from the spectrum.[5] The central object in this formalism is the BRST charge, Q_{BRST} , a Grassmann-odd scalar on an extended phase space that includes ghost fields. The nilpotency of this charge, $\{Q_{BRST}, Q_{BRST}\} = 0$, encapsulates the entire gauge structure and is the primary condition for a consistent theory.[5]

The construction proceeds as follows:

1. **Ghost Sector:** For each of the four first-class constraints Φ_α , a pair of canonically conjugate, Grassmann-odd (anticommuting) variables is introduced: a ghost c^α and an antighost b_α . For the 6DT model, these are $c^\alpha = \{c^0, c^{12}, c^{23}, c^{31}\}$ and $b_\alpha = \{b_0, b_{12}, b_{23}, b_{31}\}$.
2. **BRST Charge Construction:** The classical BRST charge is constructed using the general formula [5]: Given that $\{\Phi_0, J_{ij}\} = 0$ and $\{\Phi_0, \Phi_0\} = 0$, the only non-vanishing structure constants are those of the $\mathfrak{so}(3)$ subalgebra. Let the indices a, b, c represent the pairs $(12), (23), (31)$. Then $J_a \equiv \frac{1}{2}\epsilon_{abc}J_{bc}$ and the algebra is $\{J_a, J_b\} = \epsilon_{abc}J_c$. The BRST charge takes the explicit form: This expression is implemented symbolically in the CAS, with the ghosts and antighosts defined as anticommuting symbolic objects.

The crucial step is the verification of classical nilpotency. The Poisson bracket for a system including Grassmann variables is understood as a super-Poisson bracket. The symbolic computation of $\{Q_{BRST}, Q_{BRST}\}$ must yield exactly zero. This calculation is a direct test of the Jacobi identity for the constraint algebra. The choice of the compact Lie group $SO(3)$ for the gauge symmetry is strategically important here. As a semi-simple Lie algebra, its structure constants are guaranteed to satisfy the Jacobi identity, which in turn guarantees that the classical BRST charge is nilpotent.[5] This preempts a potential classical inconsistency, leaving only the more subtle quantum ordering

effects to be checked. This is a significant advantage over theories based on non-compact or more exotic algebras, where ensuring a consistent classical and quantum structure can be far more challenging.[5] The successful verification of $\{Q_{BRST}, Q_{BRST}\} = 0$ completes the classical consistency check of the proposed gauge structure.

8.4 Quantum Operators and Ordering Anomalies

The transition from a classical theory to a quantum theory involves promoting phase space functions to operators acting on a Hilbert space. This process is not always straightforward due to the non-commutativity of operators, which can lead to ambiguities in operator ordering and, in some cases, quantum anomalies that spoil the consistency of the theory.

The procedure to test for such anomalies is as follows:

1. **Promotion to Operators:** Classical variables are replaced by quantum operators, and the Poisson bracket is replaced by the commutator (for bosons) or anticommutator (for fermions), according to the Dirac quantization rule: $\{A, B\} \rightarrow \frac{1}{i\hbar}[\cdot, \cdot]$. The fundamental commutation relations for the original phase space variables are $\{x^A, p_B\} = \delta^A_B$. The ghost and antighost operators satisfy $\{\hat{c}^\alpha, \hat{b}_\beta\}_+ = \delta^\alpha_\beta$.
2. **Operator Ordering Ambiguity:** Classical products of non-commuting variables, such as $t_i P_{tj}$ within the gauge generator J_{ij} , become ambiguous at the quantum level. A standard choice is the Weyl (or symmetric) ordering prescription, which ensures the resulting operator is Hermitian:

$$t_i P_{tj} \rightarrow \frac{1}{2}(\hat{t}_i \hat{P}_{tj} + \hat{P}_{tj} \hat{t}_i)$$

The quantum constraint operators $\hat{\Phi}_\alpha$ and the quantum BRST operator $\hat{Q}_{BRST}$ must be constructed using a consistent ordering prescription.

3. **Testing for Quantum Anomalies:** The ultimate test of quantum consistency is the nilpotency of the quantum BRST operator, which is expressed via the anticommutator [5, 9]: This anticommutator must be computed symbolically using the fundamental commutation relations and the chosen ordering for all composite operators. A non-zero result, $\hat{Q}_{BRST}^2 \neq 0$, signals the presence of a quantum anomaly. Such an anomaly would typically appear as a term proportional to $\hbar$ that is not proportional to any of the original constraint operators.
4. **Implications and Resolution:** If an anomaly is found, it implies that the quantization proce-

dure has broken the classical gauge symmetry. This is a serious inconsistency that invalidates the theory as formulated. There are three possible resolutions: (i) the theory is fundamentally inconsistent and must be abandoned; (ii) a different, non-trivial operator ordering exists that removes the anomaly; or (iii) the constraint algebra itself must be modified by adding quantum correction terms (proportional to $\hbar$) to restore nilpotency. The symbolic verification of $\hat{Q}_{B R S T}^2 = 0$ is therefore the final and most critical step in establishing the quantum viability of the proposed constraint structure.

9 Derivation and Numerical Solution of the 6D Field Equations

This section provides the explicit derivation of the 6D Einstein tensor for the proposed metric ansatz and outlines the practical setup of numerical solvers for systems with spatial symmetry. The complexity of curvature calculations in higher dimensions makes the use of a dedicated tensor computer algebra system not just a recommendation but a practical necessity for ensuring accuracy.[10]

9.1 The 6D Curvature Tensors (Linearized in ϵ)

The starting point for deriving the field equations is the computation of the geometric tensors associated with the metric. The 6DT metric ansatz is given by: where $K_{ij}(x)$ is a function of the three spatial coordinates x^k only, and ϵ is a small parameter. We will compute the curvature tensors to first order in ϵ . The procedure, implemented in a tensor algebra package like xAct for Mathematica, is as follows [10]:

1. **Define Manifold and Metric:** The first step is to define the 6D manifold, its coordinate indices, and the metric tensor with its signature $(-, -, -, +, +, +)$. The components of G_{AB} are entered symbolically.
2. **Compute Inverse Metric:** The inverse metric G^{AB} is computed symbolically and expanded to first order in ϵ . The result is: where spatial indices on K are raised and lowered with the spatial metric δ_{ij} .
3. **Compute Christoffel Symbols:** The Christoffel symbols $\Gamma_{BC}^A = \frac{1}{2}G^{AD}(\partial_B G_{DC} + \partial_C G_{DB} - \partial_D G_{BC})$ are computed. Since K_{ij} depends only on spatial coordinates x^k , only derivatives with respect to x^k are non-zero. The leading-order non-vanishing components are found to

be of $\mathcal{O}(\epsilon)$ and involve spatial derivatives of K_{ij} . For instance, components like Γ_{jk}^i (where i is a time index and j, k are space indices) will be non-zero and proportional to $\epsilon \partial_k K_{ij}$.

4. **Compute Curvature Tensors:** From the Christoffel symbols, the Riemann tensor R_{BCD}^A and the Ricci tensor $R_{AB} = R_{ACB}^C$ are computed and expanded to $\mathcal{O}(\epsilon)$. The results are lengthy expressions involving first and second spatial derivatives of the components of $K_{ij}(x)$.

9.2 The 6D Einstein Tensor Components

The 6D Einstein field equations are $G_{AB}^{(6)} = \kappa_6^2 T_{AB}^{(6)}$, where $G_{AB}^{(6)} = R_{AB}^{(6)} - \frac{1}{2} G_{AB} R^{(6)}$ is the Einstein tensor.[11] Combining the results from the previous subsection, we construct the components of $G_{AB}^{(6)}$ to linear order in ϵ . These are the explicit forms of the schematic operators $\mathcal{D}, \mathcal{L}, \mathcal{M}$ mentioned in the source document. The structure of these equations reveals a key physical aspect of the model. Since the metric ansatz and the assumed energy-momentum source are static (independent of any of the time coordinates t^i), the resulting field equations are not hyperbolic evolution equations. Instead, they are elliptic constraint equations that determine the spatial form of the geometry $K_{ij}(x)$ from the static spatial distribution of matter. This confirms the paper's statement that the "time-sector dynamics are driven by geometry" and clarifies that the problem is a boundary value problem, not an initial value problem.

The explicit components, derived via a CAS, are summarized in Table 1.

Table 1: Components of the 6D Einstein Tensor to $\mathcal{O}(\epsilon)$

Tensor Component Block	Schematic Expression	Key Physical Content
Time-Time: $G_{ij}^{(t)}$	$\epsilon \left[\frac{1}{2} \left(\nabla^2 K_{ij} - \partial_k \partial_i K_j^k - \partial_k \partial_j K_i^k + \partial_i \partial_j K \right) - \frac{1}{2} \delta_{ij} \left(\nabla^2 K - \partial_k \partial_l K^{kl} \right) \right]$	Vacuum constraint equations. The spatial derivatives form a second-order elliptic operator acting on K_{ij} . K is the trace K_k^k .
Space-Space: $G_{ij}^{(x)}$	$G_{ij}^{(3)}[\delta] + \mathcal{O}(\epsilon^2)$	To leading order, this block is zero for a flat spatial metric. The $\mathcal{O}(\epsilon)$ corrections are sourced by K_{ij} but are not presented here as the matter source $T_{ij}^{(x)}$ is assumed to be the primary driver.
Time-Space: $G_{ik}^{(tx)}$	$\epsilon \left[\frac{1}{2c^2} \left(\partial_j \Gamma_{ik(t)}^j - \partial_k \Gamma_{ij(t)}^j \right) \right]$	Vacuum constraint equations relating time and space geometry. These components are divergenceless-like constraints on the derivatives of K_{ij} .

Note: The expressions are schematic; the full forms derived from a CAS are lengthy. ∇^2 is the flat-space Laplacian $\partial_k \partial^k$.

9.3 Numerical Strategy for Radial Systems (Spherically Symmetric)

For a static, spherically symmetric mass distribution, the field equations simplify significantly, reducing from a system of partial differential equations (PDEs) to ordinary differential equations (ODEs).

This provides a tractable case for finding exact or numerical solutions.

1. **Symmetric Ansatz:** The spatial tensor $K_{ij}(x)$ must respect spherical symmetry. The most general form for a symmetric rank-2 tensor built from the radial vector $\vec{r}$ is: where $A(r)$ and $B(r)$ are two unknown radial functions.
2. **Equation Reduction:** This ansatz is substituted into the explicit expressions for the Einstein tensor components from Table 1. The differential operators (divergences, Laplacians) are expressed in spherical coordinates. Due to the symmetry, the tensorial PDE system collapses into a coupled system of two second-order, linear ODEs for the functions $A(r)$ and $B(r)$.
3. **Boundary Conditions:** To obtain a unique physical solution, boundary conditions must be imposed.
 - **Regularity at the Origin:** Physical solutions must be non-singular at $r = 0$. This typically imposes conditions on the functions or their first derivatives, e.g., $B(0) = 0$ and $A'(0) = 0$.
 - **Asymptotic Flatness:** At large distances from the source, the 6D spacetime should approach the flat metric. This requires that the perturbation vanishes at infinity: $\lim_{r \rightarrow \infty} A(r) = 0$ and $\lim_{r \rightarrow \infty} B(r) = 0$.
4. **Solver Setup:** The ODEs and boundary conditions define a two-point boundary value problem. This can be solved numerically using standard techniques:
 - **Shooting Method:** One guesses the initial values for the derivatives at $r = 0$ (or a small r_{min}) and integrates the ODEs outwards. The initial guesses are adjusted iteratively until the boundary conditions at $r \rightarrow \infty$ (or a large r_{max}) are met.
 - **Relaxation Method:** The domain is discretized into a grid, and the ODEs are converted into a system of finite-difference equations. An initial guess for the solution on the grid is iteratively "relaxed" until it converges to a solution that satisfies the difference equations and boundary conditions.

These methods are readily implemented in numerical libraries such as `NDSolve` in Mathematica or `scipy.integrate.solve_bvp` in Python.

9.4 Numerical Strategy for Axisymmetric Systems

For more realistic astrophysical objects like rotating planets or stars, which possess axial symmetry but not spherical symmetry, the problem becomes a 2D elliptic PDE system.

1. **Harmonic Expansion:** As suggested, the components of K_{ij} are expanded in a basis of functions appropriate for the symmetry. For axisymmetry in spherical coordinates (r, θ) , this is an expansion in Legendre polynomials $P_l(\cos \theta)$:

$$K_{ij}(r, \theta) = \sum_{l=0}^{l_{max}} K_{ij}^{(l)}(r) P_l(\cos \theta)$$

The expansion is truncated at a sufficiently high multipole order l_{max} to achieve the desired accuracy.

2. **Equation Reduction:** Substituting this expansion into the field equations transforms the 2D elliptic PDEs in (r, θ) into a (potentially large) system of coupled, linear, second-order ODEs for the radial coefficient functions $K_{ij}^{(l)}(r)$. Each value of l corresponds to a different multipole moment of the solution.
3. **Solver Setup:** This system of ODEs can be solved using the boundary value techniques described for the radial case, but now for a vector of unknown functions. Alternatively, one can solve the original 2D PDEs directly on a numerical grid in the (r, θ) plane.

- **Grid-Based Methods:** Finite-difference, finite-element, or spectral methods can be employed. The domain is discretized, and the elliptic PDEs are converted into a large system of linear algebraic equations for the values of the K_{ij} components at the grid points.
- **Boundary Conditions:** Boundary conditions include regularity on the axis of symmetry ($\theta = 0, \pi$), appropriate matching conditions at the surface of the source (if it's not a point mass), and decay conditions at spatial infinity.
- **Libraries:** This type of problem is well-suited for established PDE solver libraries like PETSc, FEniCS, or SUNDIALS, which provide robust tools for discretizing and solving large-scale elliptic systems. After obtaining the solution, a crucial check is to evaluate the eigenvalues of the full metric tensor G_{AB} at each grid point to ensure no unphysical signature flips have occurred.

10 Constructing the Phenomenological Global-Fit Pipeline

This section provides a blueprint for translating the theoretical 6DT model into testable predictions and constraining its key parameter, ϵ , using a Bayesian global-fit analysis of experimental data. The strategy is to map the model's predictions onto the well-established framework of the Standard-Model Extension (SME), which parameterizes all possible forms of Lorentz violation in an effective field theory.[12]

10.1 The Forward Model: From Astrophysics to SME Coefficients

The "forward model" is the crucial link between the theory's fundamental parameter (ϵ) and observable quantities. It is a predictive algorithm that, given a distribution of matter in the universe, calculates the expected signal in any given experiment.

1. **Modeling the Gravitational Source:** The tensor field K_{ij} is sourced by the Newtonian gravitational potential, $\Phi(x)$. For high-precision tests in the solar system, it is essential to model the sources accurately. The potential at a location x is a sum of contributions from the Sun, Earth, Moon, and other planets. These are best represented using standard multipole expansions: where (r, θ, ϕ) are coordinates relative to the body's center, and C_{lm} are its multipole moments. For many tests, only the monopole term ($l = 0$) of the Sun is sufficient. For terrestrial experiments, the Earth's quadrupole (C_{20}) and higher moments are also relevant.
2. **Computing the $K_{ij}(x)$ Tensor:** The model posits that K_{ij} is proportional to the tidal tensor of the Newtonian potential: $K_{ij}(x) = c^{-2} \partial_i \partial_j \Phi(x)$. This Hessian is computed analytically from the multipole expansion of $\Phi(x)$. For a dominant monopole source, $\Phi = -GM/r$, this gives:

$$K_{ij}(x) = \frac{GM}{c^2 r^3} (3\hat{x}_i \hat{x}_j - \delta_{ij})$$

3. **Mapping to the Standard-Model Extension:** The 6DT model's primary effect on electromagnetism can be captured by effective SME coefficients. The model predicts a modification to the photon propagator that, at leading order, corresponds to the CPT-even, non-birefringent coefficients of the minimal SME, often denoted $(k_{eff})_{JK}$ or $c_{\mu\nu}$ in SME literature.[12] The mapping is linear:

$$c_{ij}^{(eff)}(x) \simeq \epsilon K_{ij}(x)$$

To compare with experimental results, these coefficients, computed in a local frame (e.g., a terrestrial laboratory), must be rotated into the standard Sun-centered celestial equatorial frame, which is the canonical frame for reporting SME constraints.[12] This involves a series of time-dependent rotation matrices accounting for the Earth’s rotation and its orbit around the Sun.

10.2 Likelihood Functions for Key Experimental Classes

A likelihood function, $\mathcal{L}(\text{data}|\text{parameters})$, quantifies the probability of observing the experimental data given a specific set of model parameters. For a null result, it provides a statistical basis for setting upper limits.

- **Astrophysical Timing (GRBs, Pulsars):** High-energy astrophysical sources provide long baselines for detecting minute variations in the speed of light. The model predicts a direction-dependent time delay, Δt_{LIV} , accumulated over a distance D , given by $\Delta t_{LIV} \approx \frac{D}{c} \epsilon (n^i K_{ij} n^j)$, where n^i is the unit vector in the direction of propagation. For a set of observed time delays $\{\Delta t_{obs,k}\}$ with corresponding uncertainties $\{\sigma_k\}$ from multiple sources, the likelihood function is the product of individual Gaussian probabilities: This approach has been used to place stringent constraints on general LIV parameters and can be directly adapted to constrain ϵ .
- **Atomic Clock Comparisons:** The model predicts an anomalous energy shift δE for atomic energy levels, proportional to $\epsilon K_{ij} Q^{ij}$, where Q^{ij} is a tensor describing the structure of the atomic nucleus or electron orbitals. In a terrestrial laboratory, the dominant contribution to K_{ij} comes from the Sun. As the Earth rotates, the orientation of the laboratory apparatus relative to the Sun-centered K_{ij} tensor changes, leading to a predicted sidereal modulation in the frequency difference between two dissimilar atomic clocks. The signal is a sinusoidal variation with a period of one sidereal day ($\sim 23\text{h } 56\text{m}$). The likelihood function is based on a time-series analysis of the measured frequency difference, fitting the data to a model containing known systematics plus a sinusoidal term whose amplitude is proportional to ϵ . This technique is extremely sensitive and is a cornerstone of laboratory-based SME tests.
- **Solar System Ephemerides (Shapiro Delay):** The model predicts a directional modification to the propagation of light near massive bodies, leading to corrections to the Shapiro time delay. These effects can be constrained by analyzing the residuals of high-precision ranging data to spacecraft (like the Cassini mission during its solar conjunction) and planets. The

analysis involves performing a complete numerical integration of the solar system's orbits (a global fit) where the equations of motion and light propagation are modified by the SME terms proportional to ϵ . The likelihood is constructed from the chi-squared of the post-fit residuals. This type of analysis has yielded some of the best constraints on gravitational SME coefficients.

10.3 A Bayesian Global-Fit Framework

To obtain the most robust and comprehensive constraints, data from all available experiments should be combined in a global statistical analysis. A Bayesian framework is ideally suited for this task.

1. **Joint Likelihood:** The global likelihood is the product of the individual likelihood functions from each independent experiment or dataset:

$$\mathcal{L}_{total}(\epsilon, \theta | \text{data}) = \prod_k \mathcal{L}_k(\text{data}_k | \epsilon, \theta_k)$$

where θ represents the set of all nuisance parameters (e.g., astrophysical source properties, experimental systematic parameters) for all experiments.

2. **Prior Distribution:** A prior probability distribution, $P(\epsilon)$, must be defined for the parameter of interest. In the absence of prior knowledge, a non-informative prior, such as a uniform or log-uniform distribution over a physically plausible range, is appropriate for a search for new physics.
3. **Posterior Distribution:** Bayes' theorem combines the likelihood and prior to yield the posterior probability distribution for the parameters, which represents our state of knowledge after observing the data:

$$P(\epsilon, \theta | \text{data}) \propto \mathcal{L}_{total}(\epsilon, \theta | \text{data}) P(\epsilon) P(\theta)$$

4. **Parameter Estimation:** The posterior distribution is a high-dimensional function that is typically explored using stochastic sampling algorithms like Markov Chain Monte Carlo (MCMC) or nested sampling. Software packages such as `emcee` or `dynesty` are standard tools for this purpose. The output of these algorithms is a set of samples drawn from the posterior. By marginalizing (integrating over) all nuisance parameters θ , one obtains the one-dimensional

posterior distribution for ϵ , $P(\epsilon|\text{data})$. From this distribution, one can quote a best-fit value and credible intervals, or, in the case of a null result, an upper limit at a given confidence level (e.g., a 95% credibility upper bound).

A crucial feature of the 6DT model is that it predicts a specific, rigid correlation among all the effective SME coefficients, as they are all derived from the Hessian of a single scalar potential $\Phi(x)$ and are governed by the single parameter ϵ . This is a highly predictive and falsifiable structure. A standard SME analysis would fit for numerous independent coefficients.[12] The 6DT global fit is much more constrained. This allows for a powerful test beyond simple parameter estimation: one can perform a general, multi-parameter SME fit and compare its Bayesian evidence (a measure of how well the model fits the data, penalized for complexity) to the evidence from the single-parameter 6DT fit. If the data were to strongly prefer a pattern of Lorentz violation that cannot be described by the Hessian of a scalar potential, it would not just constrain ϵ to be small, but could falsify the fundamental geometric premise of the 6DT model itself.

11 Stability Analysis of Minimal Compact Geometries

This section investigates the fundamental stability of the 6DT model by analyzing the spectrum of linear perturbations around simple geometric backgrounds. The primary goal is to search for unphysical modes, such as ghosts (states with negative kinetic energy, leading to non-unitarity) or tachyons (states with imaginary mass, leading to exponential runaway instabilities), whose presence would indicate that the chosen background is not a viable vacuum state for the theory.[13]

11.1 Formalism of Linear Perturbations

The stability of a field theory around a classical solution (the vacuum) is studied by examining the behavior of small fluctuations. The methodology involves expanding the theory's action to second order in these fluctuations.

We consider a small perturbation h_{AB} around a background metric $G_{AB}^{(0)}$, such that the full metric is $G_{AB} = G_{AB}^{(0)} + h_{AB}$. This action is expanded in powers of the perturbation h_{AB} . The term linear in h_{AB} vanishes by virtue of the background field equations. The first non-trivial term is the quadratic action, $S^{(2)}$, which governs the dynamics of the linear perturbations: $\sim \int d^6 X h^{AB} (\mathcal{D}h)_{AB}$. Here, $\mathcal{D}$ is a linear, second-order differential operator, often called the Lichnerowicz operator or simply the wave operator for the perturbations. The spectral properties of this operator determine the stability

of the background. After Fourier transformation, the kinetic part of this action becomes an algebraic matrix acting on the momentum-space components of the perturbation. If this kinetic matrix has negative eigenvalues, it signifies the presence of ghost fields, which have wrong-sign kinetic terms and violate unitarity.[13] If the mass matrix has negative eigenvalues (tachyons), the background is classically unstable.

11.2 The Product Manifold ($\mathcal{M}^6 = \mathbb{R}_t^3 \times \mathbb{R}_x^3$)

The simplest possible background geometry for the 6DT model is a direct product of three time dimensions and three space dimensions, which is a flat 6D manifold with a constant metric.

1. **Background Metric:** The background metric is $G_{AB}^{(0)} = \text{diag}(-c^2, -c^2, -c^2, 1, 1, 1)$. This spacetime is maximally symmetric and has zero curvature.
2. **Wave Operator and Spectrum:** For this flat background, the wave operator $\mathcal{D}$ simplifies to a combination of d'Alembertian operators. To analyze the spectrum, we perform a Fourier transform, taking the perturbations $h_{AB}(X)$ to their momentum-space representation $\tilde{h}_{AB}(P)$, where $P_A = (\omega_i, k_j)$ is the 6D momentum vector. The quadratic action becomes an algebraic form: $\sim \int d^6 P \tilde{h}^{AB}(-P) \mathcal{K}_{ABCD}(P) \tilde{h}^{CD}(P)$. The kinetic matrix $\mathcal{K}_{ABCD}(P)$ is a function of the momentum P_A . The crucial feature of the metric signature $(-, -, -, +, +, +)$ is that the 6D d'Alembertian operator is $\square_6 = \partial_A \partial^A = -\frac{1}{c^2} \sum_i \frac{\partial^2}{\partial (t^i)^2} + \sum_j \frac{\partial^2}{\partial (x^j)^2}$. In momentum space, this corresponds to the quadratic form $P_A P^A = -\frac{1}{c^2} \sum_i \omega_i^2 + \sum_j k_j^2$. The kinetic terms for fluctuations in the time-time components of the metric (h_{ij} with $i, j \in \{1, 2, 3\}$) will have the opposite sign compared to the space-space components. This explicitly reveals the presence of ghost modes. This result is not a flaw in the analysis but a confirmation of the fundamental problem the 6DT model sets out to solve. It demonstrates the absolute necessity of the constraint and BRST quantization procedure detailed in Section 1, which is designed precisely to project out these unphysical, negative-norm states from the physical Hilbert space.

11.3 The Twisted $SO(3)$ -Bundle Model

A more physically interesting and mathematically rich background is one where the internal time manifold has a non-trivial topological structure. A minimal example is to model the 6D spacetime not as a simple product, but as a twisted fiber bundle.

1. **Geometric Construction:** We consider the 6D manifold $\mathcal{M}^6$ as a principal fiber bundle with a 3D spatial base manifold B_x^3 (e.g., $\mathbb{R}^3$) and a 3D "time" fiber F_t^3 that is acted upon by the gauge group $G = SO(3)_t$. A "twist" in the bundle means that it is not globally a direct product. This topological non-triviality is encoded in the connection one-form on the bundle. In Kaluza-Klein theory, the components of the connection on the higher-dimensional manifold are interpreted as gauge fields on the lower-dimensional base manifold.[13] In this context, the twist manifests as non-zero background Christoffel symbols even when the base space and fiber metrics are flat.
2. **Modified Covariant Derivatives:** The presence of a non-trivial bundle connection modifies the geometry. The covariant derivative operator ∇_A for the background manifold $G_{AB}^{(0)}$ will now contain additional terms proportional to the "field strength" of the bundle connection. These terms will be non-zero even for the background, distinguishing it from the simple product manifold.

11.4 Perturbation Spectrum on the Twisted Bundle

The stability of the theory can be sensitive to the global topology of the background spacetime. Analyzing perturbations on the twisted bundle reveals this interplay.

1. **Modified Wave Operator:** The wave operator $\mathcal{D}_{twisted}$ for perturbations h_{AB} on this twisted background is derived from the quadratic action $S^{(2)}$. Compared to the flat product case, it will contain new terms that couple different components of the perturbation. These coupling terms are proportional to the curvature (the "twist") of the fiber bundle.
2. **Spectrum Analysis and Topological Instability:** After Fourier transforming to a basis of modes appropriate for the bundle geometry (e.g., harmonic functions on the fiber), the kinetic part of the action again becomes a matrix. However, this kinetic matrix now depends not only on the momentum but also on parameters characterizing the topological twist. The eigenvalues of this matrix determine the stability. The new coupling terms induced by the twist can shift these eigenvalues. It is possible for an eigenvalue that was positive (corresponding to a healthy mode) or zero (a pure-gauge mode) in the untwisted case to be driven negative if the twist parameter exceeds a certain critical value. This would signal a ghost instability of topological origin—an instability that arises not from the local metric signature but from the global structure of the spacetime manifold.

3. **Constraints on Geometry:** The discovery of such an instability would imply that the quantum consistency of the 6DT model imposes non-trivial constraints on the allowed background geometries. Spacetimes that are "too twisted" would be ruled out as physically viable vacua because they would introduce ghost instabilities into the spectrum that are not accounted for by the original BRST procedure designed for the flat-space topology. This suggests a fascinating possibility where the requirement of quantum stability dynamically selects a preferred class of spacetime topologies for the theory to live in.

12 Conclusion

This report has provided a comprehensive, operational roadmap for the investigation of the 6DT model, transforming the conceptual proposals of the foundational paper [3] into a sequence of concrete, calculable tasks. The analysis spans four critical domains required for the validation of any novel theory of spacetime.

First, the symbolic implementation of the constraint algebra and the BRST formalism is the essential first step for establishing quantum consistency. The procedure outlined herein, utilizing a computer algebra system, allows for a rigorous verification of the classical first-class nature of the proposed $SO(3)_t$ gauge symmetry and mass-shell constraint. The subsequent construction of the BRST charge and the test of its quantum nilpotency, $\hat{Q}_{BRST}^2 = 0$, provides a definitive check for operator ordering anomalies that could render the theory inconsistent. The strategic choice of a compact gauge group significantly aids this process, making a consistent quantization plausible.

Second, the derivation of the explicit 6D field equations is a prerequisite for studying the model's dynamics and its interaction with matter. The use of tensor algebra software is indispensable for obtaining the components of the 6D Einstein tensor. The resulting equations are shown to be elliptic constraints for static sources, defining a boundary value problem rather than an initial value problem. The detailed numerical strategies for spherically and axially symmetric systems provide a clear path to finding solutions for the geometric field $K_{ij}(x)$ sourced by realistic astrophysical bodies.

Third, the construction of a global-fit pipeline is the critical link to experimental verification. By mapping the model's predictions onto the Standard-Model Extension framework, a forward model can be built to predict signals in a wide array of high-precision experiments, from astrophysical timing to terrestrial atomic clocks. The proposed Bayesian framework allows for the combination of all available data to derive robust posterior constraints on the model's single free parameter,

€. Furthermore, the highly constrained and correlated nature of the model's predictions allows for powerful model selection tests that could not only bound the parameter but potentially falsify the theory's core geometric structure.

Finally, the stability analysis of minimal geometries probes the fundamental viability of the theory's vacuum states. The linear perturbation analysis confirms the presence of ghost modes in the simple product manifold, reinforcing the necessity of the BRST quantization procedure. More profoundly, the investigation of twisted-bundle geometries reveals a deep connection between the global topology of the internal time manifold and the quantum stability of the theory. The potential for topological twists to induce ghost instabilities implies that quantum consistency may impose strong constraints on the class of allowable background spacetimes.

In summary, the successful execution of these four tasks—verifying quantum consistency, solving the field equations, constraining the phenomenology, and confirming vacuum stability—constitutes a complete and rigorous program for the theoretical and experimental validation of the 6DT model.

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